Short impulse-propagation in inhomogeneous plasma

ORSOLYA E. FERENCZ

Space Research Group, Eötvös University, Geophysical Department

spacerg@sas.elte.hu

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In this paper the problem of real impulse-propagation in arbitrarily inhomogeneous media will be presented on a fundamentally new, general, theoretical way. The general problem of wave-propagation of monochromatic signals in inhomogeneous media was enlightened in [1]. The former theoretical models for spatial inhomogeneities have some errors regarding the structure of the resultant signal originated from backward and forward propagating parts. The application of the Method of Inhomogeneous Basic Modes (MIBM) and the complete full-wave solution of arbitrarily shaped non-monochromatic plane-waves in plasmas made it possible to obtain a better description of the problem, on a fully analytical way, directly from Maxwell's equations. The model investigated in this paper is inhomogeneous of arbitrary order (while the wave-pattern can exist), anisotropic (magnetized), linear, cold plasma, in which the gradient of the one-dimensional spatial inhomogeneity is parallel to the direction of propagation.

The traditional theoretical descriptions of a monochromatic electromagnetic signal propagating in an inhomogeneous medium – e.g. eikonal-equation, W.K.B. method, generalized propagation-vector, application of Airy-functions in solving the Stokes equation etc. – have a common fundamental inaccurate assumption relating the physical concept of the structure of the signal. The model of the solution in these approaches is an additional sum of the different signal-parts, propagating forward or backward (scattered), which parts are supposed solutions of Maxwell's equations independently of each other. However, this way of description does not make it possible to recognize the influence of the real energetic coupling between these signal-parts, as for the real full-wave solution of Maxwell's equations always has to contain all the existing modes simultaneously.

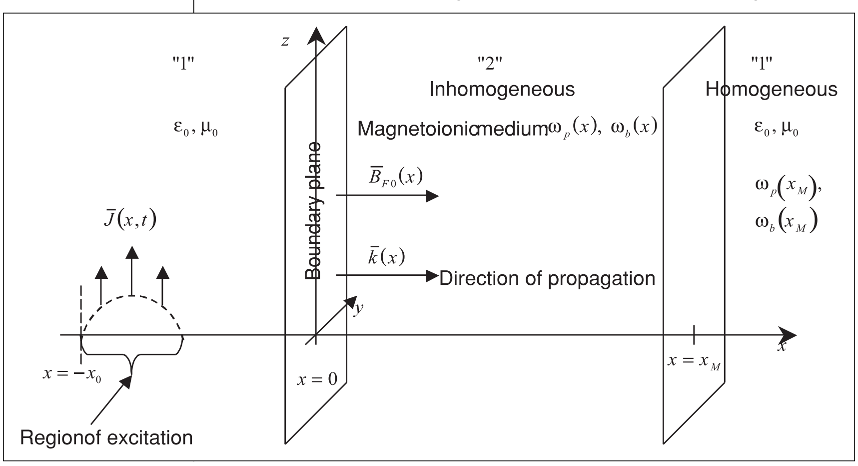
It is a well-known fact that the additional sum of solutions has to fulfill the original equations, if a linear differential equation-system has some solutions. Nevertheless, it does not mean that the parts of the solution fulfilling the equations – separated from each other by application of different theoretical points of view – would automatically be solutions of the full equation-system.

The resultant and existing signal is a solution of Maxwell's equations, but its parts (the backward and forward propagating signal-parts) are not.

Moreover, as it can be seen in the coupled W.K.B. philosophy, the influence of the reflection from the spatial inhomogeneity is neglected in the assumed signal-form in the traditional ways of thinking. The reflected signal-form is created on the same way – as if an independent mirror-source would exist –, but the coupling is supposed to be determined by a simple addition of these signals, although this influence was eliminated in their creation. A correct mathematical analysis of this situation will yield a contradiction for the resultant field, because the curl-equations become automatically over-determined neglecting the energetic coupling between the different signal parts.

A further theoretical problem, unanswered up to now, is originated from the fact, that a real physical signal excited by an impulse -– or switching on-off transient – is always non-monochromatic and its description is not enough accurate by superposition of monochromatic signals [2,3,4,5]. With other words, the problem of arbitrarily shaped signals does not make it possible to assume any exp(jωt)-type starting form of the solution at the beginning of the derivation of Maxwell's equations.

Fig. 1.
Model for longitudinal propagation.
Medium "1" is homogenous and "2" is inhomogeneous.



1. Derivation of the full-wave solution

The applied model and method

The example, by that the theoretical solution will be presented here, is a linear, time-invariant, anisotropic, cold plasma. The source is an arbitrarily shaped plane-wave (e.g. an "impulse plane"), the direction of the propagation and the gradient of the one-dimensional inhomogeneity are parallel to the superimposing magnetic field. The order of the magnitude of the inhomogeneity is not restricted, except the precondition that the wave front can be defined *(Fig. 1.)*

As it was briefly mentioned above, the main question is how to handle the continuous generation of the reflected (scattered) signal and the energetic coupling between the existing signal-parts during the propagation.

The Method of Inhomogeneous Basic Modes (MIBM) is well applicable for such problems. The philosophy of this method [6] considers the form of the final solution to be determined as a sum of the so-called basic modes, which are not the full solutions of Maxwell's equations independently in themselves.

$$\overline{G}(\overline{r},t)=\sum_i G_i(\overline{r},t) \qquad (1)$$

where G = *E, D, H, B* and *i* is the number of the existing modes.

Substituting these basic modes into Maxwell's equations those can be disintegrated into two groups. One of them is valid even in a simple homogeneous medium; the other (the group of so-called coupling equations) characterizes the influence of the inhomogeneous medium. Boundary conditions remain as unknown variables in the full form of Maxwell's equations.

The final form of the solution can be determined by solving the coupling equations and describing these initial values.

Definition of the basic modes and the boundary conditions

For a suitable choice of the inhomogeneous basic modes it is necessary to take into consideration, that the solution must lead back to the known one valid in a homogeneous medium. (With other words, the solution for inhomogeneous case has to be a mathematical generalization of the homogeneous results.)

As a first step, the form of the solution excited by an arbitrarily shaped non-monochromatic plane-wave will be presented in homogeneous plasma.

The detailed mathematical derivation in the case of different plasma models can be found in [7,8]. If the signal to be investigated is non-harmonic, the exp(jωt) form or its superposition cannot be applied during the derivation of Maxwell's equations. The form to be determined remains open up to the final steps.

Apart from the detailed presentation (that would lead far beyond the scope of this work) the starting equations are

$$\overline{\nabla}\,'\ \overline{H}_2=\overline{J}+\varepsilon_0\frac{\partial\overline{E}_2}{\partial t},\qquad m\frac{\partial\overline{v}}{\partial t}=q\left(\overline{E}+\overline{v}\times\overline{B}_{F0}\right),$$
$$\overline{J}=qN\overline{v}, \qquad (2)$$
$$\overline{\nabla}\,'\ \overline{E}_2=-\mu_0\frac{\partial\overline{H}_2}{\partial t},\qquad \overline{\nabla}\bullet\overline{J}+\frac{\partial\rho}{\partial t}=0,$$

where ε_0 and μ_0 the permittivity and permeability in vacuum, respectively. The electron density of the plasma is *N*, the superimposing magnetic field is B_{F0}, *m* and *q* = –*e* are the mass and charge of an electron, $\overline{v}$ is the velocity of the electrons.

The plasma- and the gyro-frequencies are

$$\omega_b=\frac{eB_{F0}}{m}\qquad \omega_p^2=\frac{q^2N}{\varepsilon_0 m}\equiv\frac{e^2N}{\varepsilon_0 m}. \qquad (3)$$

By some mathematical transformation of (2), the following differential equations can be obtained for the propagating field (4)

$$\frac{\partial^2E_y}{\partial x^2}=\frac{1}{c^2}\left\{\omega_p^2\int_0^t\frac{\partial E_y}{\partial\tau}\cos\omega_b(t-\tau)\cdot d\tau-\omega_b\omega_p^2\int_0^t E_z\cos\omega_b(t-\tau)\cdot d\tau+\frac{\partial^2E_y}{\partial t^2}\right\},$$
$$\frac{\partial^2E_z}{\partial x^2}=\frac{1}{c^2}\left\{\omega_p^2\int_0^t\frac{\partial E_z}{\partial\tau}\cos\omega_b(t-\tau)\cdot d\tau+\omega_b\omega_p^2\int_0^t E_y\cos\omega_b(t-\tau)\cdot d\tau+\frac{\partial^2E_y}{\partial t^2}\right\}.$$

As the form of the solution is completely unknown, (4) is not solvable in time-space domain. Therefore, it is necessary to apply the Laplace-transformation of the equations. This transformation takes into account the transient behavior of the signal.

Using the Laplace-transformation for (4) according to time and space, the unknown field components become separable. However, the application of this transformation makes it necessary to introduce initial conditions (boundary conditions) regarding the field, which will deliver the relation between the excitation and the signal propagating in the plasma.

By a suitable choice of the model structure, only two initial conditions or their Laplace-transformed forms will remain (s = jω)

$$E_z(x=0,t)\xrightarrow[s=j\omega]{L}e_{iz0t}(\omega)=A_i(\omega)$$
$$\left.\frac{\partial E_z(x,t)}{\partial x}\right|_{x=0}\longrightarrow e'_{iz0t}(\omega)=B_i(\omega) \qquad (5)$$

Solving the transformed forms of (4), the following formulas can be obtained for the propagation factor and the spectrum of the field (6)

$$\sum_{i=2}E_{yi}(\omega)=\frac{1}{4}\left\{-\left[\frac{B_1(\omega)}{k(\omega)}-jA_1(\omega)\right]e^{-jk(\omega)x}+\left[\frac{B_2(\omega)}{k(\omega)}+jA_2(\omega)\right]e^{jk(\omega)x}\right\},$$
$$\sum_{i=2}E_{zi}(\omega)=\frac{1}{4}\left\{j\left[\frac{B_1(\omega)}{k(\omega)}-jA_1(\omega)\right]e^{-jk(\omega)x}-j\left[\frac{B_2(\omega)}{k(\omega)}+jA_2(\omega)\right]e^{jk(\omega)x}\right\},$$

where *i* = 1 is the forward propagating and *i* = 2 is the reflected signal-part, and

$$k(\omega)=\frac{1}{c}\sqrt{\frac{\omega\omega_b\omega_p^2+\omega^2\left(\omega_p^2+\omega_b^2-\omega^2\right)}{\omega_b^2-\omega^2}} \qquad (7)$$

The full-wave solution from (6) in homogeneous plasma can be found in [7,8]. This paper does not deal with these details further.

The amplitude functions of (6) contain the unknown boundary conditions that represent the connection to the excitation (or with other words, the previous state of the signal).

In the definition of the inhomogeneous basic modes, (6) is well applicable with some considerations, not forgetting the fact, that the basic modes are not solutions of the problem (of Maxwell's equations) in themselves but they make it possible to obtain a full, closed form description.

As the investigated medium is inhomogeneous, the constitutional parameters depend on space, which would mean $\bar{\varepsilon}(\bar{r})$ or at least $\bar{\varepsilon}(x)$ in the case of a monochromatic signal. However, in the case of arbitrarily shaped signals it is impossible to define the closed form of the constitutional parameters, because there is no supposed sinusoidal waveform. The only open way is to give a first assumption for a space-depending form of the "propagation factor", which is not the accurate propagation factor of the inhomogeneous solution, just a first step to yield information for the real phase pattern.

A trivial form of (7) for inhomogeneous case is

$$k(x,\omega)=\frac{1}{c}\sqrt{\frac{\omega\omega_b(x)\omega_p^2(x)+\omega^2\left[\omega_p^2(x)+\omega_b^2(x)-\omega^2\right]}{\omega_b^2(x)-\omega^2}} \quad (8)$$

where

$$\omega(x)_b=\frac{eB_{F0}(x)}{m} \text{ and } \omega_p^2(x)=\frac{e^2N(x)}{\varepsilon_0 m}. \quad (9)$$

A further consideration is requested for the definition of the inhomogeneous modes.

In (6) the amplitude functions of the transformed forms contain two unknown functions that are initial conditions characteristic for the previous state of the signal. In the homogeneous case, this investigated point can be found at the boundary surface of the plasma - the entering point of the signal into the plasma. But in a spatially inhomogeneous medium the wave pattern strongly depends on the continuously varying conditions, as for the forward propagating signal excites a forward and a backward directed (reflected) signal-part in every point of the medium.

This problem can be interpreted introducing an elementarily thin "sliding boundary surface" across the plasma, as if the "entering point" of the signal-part, at which the initial conditions are valid, traveled ahead together with the propagating signal-part.

With these considerations let the inhomogeneous basic modes be defined as (10)

$$\sum_{i=2} E_{yi}(x,\omega)=\frac{1}{4}\left\{-\left[\frac{B_1(x,\omega)}{k(x,\omega)}-jA_1(x,\omega)\right]e^{-j\int k(x,\omega)dx}+\left[\frac{B_2(x,\omega)}{k(x,\omega)}+jA_2(x,\omega)\right]e^{j\int k(x,\omega)dx}\right\},$$

$$\sum_{i=2} E_{zi}(x,\omega)=\frac{1}{4}\left\{j\left[\frac{B_1(x,\omega)}{k(x,\omega)}-jA_1(x,\omega)\right]e^{-j\int k(x,\omega)dx}-j\left[\frac{B_2(x,\omega)}{k(x,\omega)}+jA_2(x,\omega)\right]e^{j\int k(x,\omega)dx}\right\},$$

$$\sum_{i=2} H_{yi}(x,\omega)=\frac{1}{4Z_0}\left\{\left[-j\frac{B_1(x,\omega)}{k_0}-\frac{k(x,\omega)A_1(x,\omega)}{k_0}\right]e^{-j\int k(x,\omega)dx}-j\left[\frac{B_2(x,\omega)}{k_0}+j\frac{k(x,\omega)A_2(x,\omega)}{k_0}\right]e^{j\int k(x,\omega)dx}\right\}, \quad (11)$$

$$\sum_{i=2} H_{zi}(x,\omega)=\frac{1}{4Z_0}\left\{\left[-\frac{B_1(x,\omega)}{k_0}+j\frac{k(x,\omega)A_1(x,\omega)}{k_0}\right]e^{-j\int k(x,\omega)dx}-\left[\frac{B_2(x,\omega)}{k_0}+j\frac{k(x,\omega)A_2(x,\omega)}{k_0}\right]e^{j\int k(x,\omega)dx}\right\}.$$

where $Z_0=\sqrt{\mu_0/\varepsilon_0}$.

Further, it must be taken into account in the investigation that the inhomogeneity is extended from $x = 0$ to $x = x_M$ spatial point, but the medium is homogeneous beyond these points ($x < 0$ and $x > x_M$). The excitation exists in the $x < 0$ half-space. Further – as this half-space is homogeneous – there is no reflected signal from $x > x_M$.

In the presence of inhomogeneity, these yield the description of the energy-coupling between the signal parts – $i = 1$ and $i = 2$ – i.e. this case; after some rearrangement we get information regarding the coupling between the propagating and reflected modes, so these are the "coupling equations".

$$\frac{\partial}{\partial x}\left[-B_1(x,\omega)+jk(x,\omega)A_1(x,\omega)\right]e^{-j\int k(x,\omega)dx}-\frac{\partial}{\partial x}\left[B_2(x,\omega)+jk(x,\omega)A_2(x,\omega)\right]e^{j\int k(x,\omega)dx}=0,$$

$$\frac{\partial}{\partial x}\left[-j\frac{B_1(x,\omega)}{k(x,\omega)}-A_1(x,\omega)\right]e^{-j\int k(x,\omega)dx}+\frac{\partial}{\partial x}\left[j\frac{B_2(x,\omega)}{k(x,\omega)}-A_2(x,\omega)\right]e^{j\int k(x,\omega)dx}=0. \quad (12)$$

Let the following simplified notations be introduced in the amplitudes

$$jk(x,\omega)A_1(x,\omega)-B_1(x,\omega)\triangleq C_1(x,\omega),$$
$$jk(x,\omega)A_2(x,\omega)-B_2(x,\omega)\triangleq C_2(x,\omega). \quad (13)$$

After some mathematical rearranging (13) results in (14)

$$\frac{\partial C_1(x,\omega)}{\partial x}=\frac{1}{2k(x,\omega)}\frac{\partial k(x,\omega)}{\partial x}\left[C_1(x,\omega)+C_2(x,\omega)e^{j2\int k(x,\omega)dx}\right],$$
$$\frac{\partial C_2(x,\omega)}{\partial x}=\frac{1}{2k(x,\omega)}\frac{\partial k(x,\omega)}{\partial x}\left[C_2(x,\omega)+C_1(x,\omega)e^{-j2\int k(x,\omega)dx}\right]$$

The solution of (14) can be obtained by successive approximation. As the first step let $C_2 = 0$ be assumed. Then

$$\frac{\partial C_1(x,\omega)}{\partial x}=\frac{1}{2k(x,\omega)}C_1(x,\omega) \quad (15)$$

further

$$\ln C_1(x,\omega)=$$
$$=\frac{1}{2}\ln k(x,\omega)+c_0(\omega) \quad (16)$$

From (16)

$$C_1(x,\omega)=C_0(\omega)\sqrt{k(x,\omega)} \quad (17)$$

Using (17) the first approximation the propagating (i = 1) field is

$$E_{z1}(x,\omega) = -\frac{j}{4}\frac{C_0(\omega)}{\sqrt{k(x,\omega)}}e^{-j\int k(x,\omega)dx} \tag{18}$$

As it can be seen in (18), the first step of the successive approximation process yields the known form of the non-coupled W.K.B. solution for weakly inhomogeneous medium. $C_0(\omega)$ delivers the connection of the solution with the excitation. It can be determined on the way presented in details in [8].

Writing back (18) into (14)

$$\frac{\partial C_2(x,\omega)}{\partial x} - \frac{1}{2k(x,\omega)}\frac{\partial k(x,\omega)}{\partial x}C_2(x,\omega) = \frac{C_0(\omega)}{2}\frac{1}{\sqrt{k(x,\omega)}}\frac{\partial k(x,\omega)}{\partial x}e^{-2j\int k(x,\omega)dx} \tag{19}$$

(19) belongs to a known type of differential equations [9], as by the following notations

$$y(x,\omega) \longleftrightarrow C_2(x,\omega),$$

$$f(x,\omega) \longleftrightarrow -\frac{1}{2k(x,\omega)}\frac{\partial k(x,\omega)}{\partial x}, \tag{20}$$

$$g(x,\omega) \longleftrightarrow \frac{C_0(\omega)}{2}\sqrt{\frac{1}{k(x,\omega)}}\frac{\partial k(x,\omega)}{\partial x}e^{-2j\int k(x,\omega)dx}. \tag{21}$$

(20-21) can be written as

$$y'(x,\omega) + y(x,\omega)f(x,\omega) = g(x,\omega) \tag{22}$$

If there is a known (ξ,η) point on the (x,y) plane – or with other words a single value of the field (the solution) is known at a given point of the medium – the solution of (22) is the following

$$y = e^{-F}\left(\eta + \int_\xi^x g(x)e^F dx\right), \text{ where } F = \int_\xi^x f(u)du \tag{23}$$

The value of the field (now C_2) is surely known at the $\xi = x_{max}$ point (at the end of the inhomogeneity), where $C_2 \equiv 0$, as for no reflected signal-part arrives from the $x > x_{max}$ homogeneous half-space. Furthermore

$$F = \int_\xi^x f(u)du = \int_\xi^x -\frac{1}{2k(u,\omega)}\frac{\partial k(u,\omega)}{\partial u}du = \frac{1}{2}\ln\frac{k(\xi,\omega)}{k(x,\omega)}$$

$$e^{-F} = \sqrt{\frac{k(x,\omega)}{k(\xi,\omega)}} \quad \text{and} \quad e^{F} = \sqrt{\frac{k(\xi,\omega)}{k(x,\omega)}} \tag{24}$$

With (23) and (24) C_2 can be obtained as (25)

$$C_2(x,\omega) = C_0(\omega)\sqrt{\frac{k(x,\omega)}{k(\xi,\omega)}}\int_\xi^x \frac{\sqrt{k(\xi,\omega)}}{2k(u,\omega)}\frac{\partial k(u,\omega)}{\partial u}e^{-2j\int_0^u k(v,\omega)dv}du$$

From (10)

$$E_{z2}(x,\omega) = -\frac{j}{4}\left[\frac{C_2(x,\omega)}{k(x,\omega)}\right]e^{j\int k(x,\omega)dx} \tag{26}$$

Substituting (25) into (26), the full-wave time-space function of the reflected field is obtained as (27)

$$E_{z2}(x,t) = -\frac{j}{8\pi}\int_{-\infty}^{\infty}\left[\frac{C_0(\omega)}{\sqrt{k(x,\omega)}}\int_\xi^x \frac{1}{2k(u,\omega)}\frac{\partial k(u,\omega)}{\partial u}e^{-2j\int_0^u k(v,\omega)dv}du\right]e^{j\left(\omega t - \int_0^x k(x,\omega)dx\right)}d\omega$$

(27) is the first approximation of the signal reflected from an arbitrarily strong inhomogeneity, during the propagation of the original signal.

As it is obvious, the more steps of the successive approximation are executed, the more accurate solutions can be obtained for the full-wave forms of the propagating and the reflected signals. The connection with the excitation is hidden in $C_0(\omega)$. On the way detailed in [7] the form of $C_0(\omega)$ coefficient originated from an arbitrarily shaped non-monochromatic signal is as follows

$$C_0(\omega) = I_{x=0}(\omega)\frac{k_0(\omega)\sqrt{k(x=0,\omega)}}{k_0(\omega) + k(x=0,\omega)} \tag{28}$$

where the starting location of the inhomogeneity is x = 0, and the transformed form of the exciting signal at the boundary surface, $I_{x=0}(\omega)$, is

$$I_{x=0}(\omega) = \int_{-\infty}^{\infty}\left[\int_{-x_0}^{0} J_0\left(l, t + \frac{l}{c}\right)dl\right]e^{-j\omega t}dt \tag{29}$$

The location of the excitation is in the $x=[-x_0,0]$ spatial interval. J_0 is the exciting current density (see equation (1.67) in [8]).

It can be seen in (27), that the backward reflected signal-part at a given point contains the integrated influence of all the reflection generated in the medium from the end of the inhomogeneity back to the investigated point (see coordinate u). This term is determined by the complete forward propagating signal from the starting point of the inhomogeneity (see coordinate v). These integrals show well the complexity of the energy-relations of the signal at a given point of the inhomogeneous medium.

The $k(x,\omega)=0$ is a well-known mathematical singularity-problem in every lossless, ideal theoretical model. In the reality, this case never occurs, as the circumstances are never ideal (presence of loss, etc.). In the case of monochromatic signals, no propagation happens at this point, but the whole energy reflects. In the case of impulses, the problem is more complex, as the singularity-problem at a given spatial point will appear only for a frequency-segment (a given frequency) in the signal, but other parts of the signal will propagate.

As the behavior of the propagation-vector becomes rapidly varying in the surroundings of the cut-off and resonance points, the W.K.B. method cannot describe the problem (as for the W.K.B. is based on the elimination of the reflection and the assumption of constant Poynting-vector, etc.).

The new method based on MIBM takes into account the reflected energy in the signal-form (see eq. 27.) by using higher ordered steps of the successive approximation (the zero-ordered approximation, identical with the W.K.B. formula, does not contain the influence of the reflection).

Therefore the new formulas approach the singularities asymptotically for each frequency, and they remain numerically manageable (some samples around the

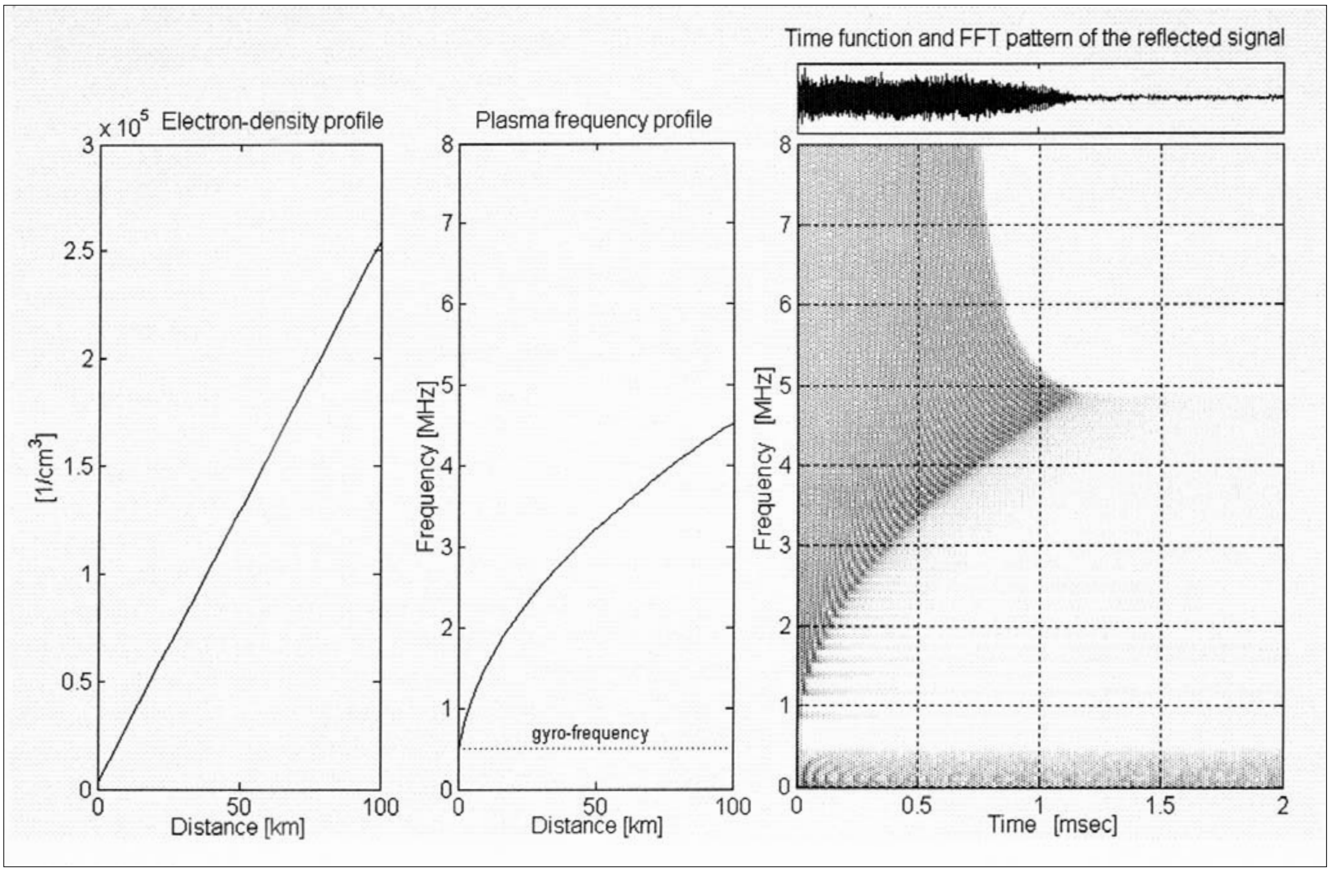


Figure. 2/a.

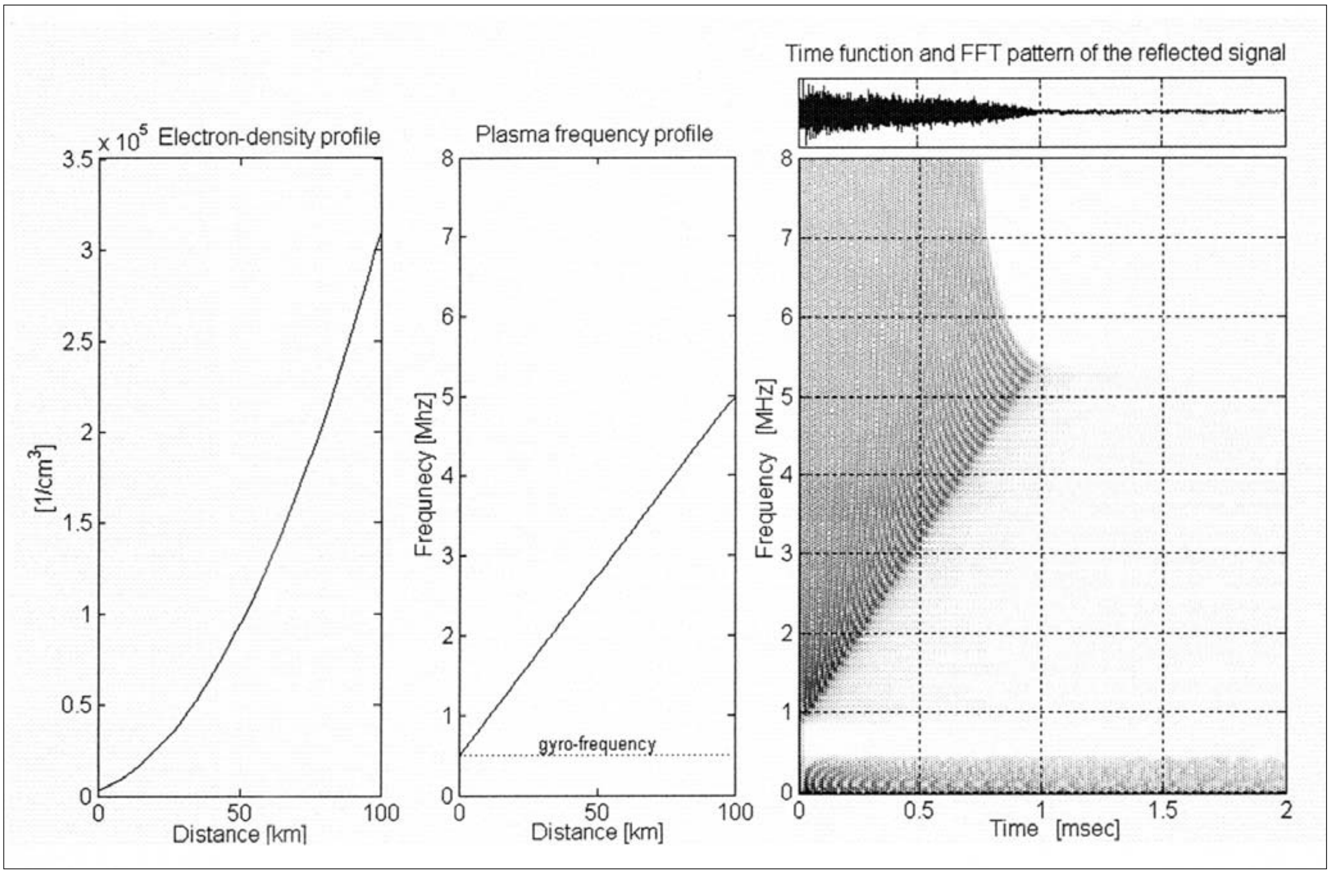


Figure. 2/b.
FFT-patterns and time-functions of calculated reflected signals for different electron-density and plasma frequency profiles.

singularities, at where the integrals become instable, are neglected from the model-calculation). This means physically that approaching this point, the reflection increases, finally up to the reflection of the forward propagating energy within a frequency-range (in which the singularity arises), so this signal-part never reaches actually this point.

Other parts of the whole signal will propagate beyond this point, up to their own singular points (if these exist somewhere).

(The comparison of the Airy-functions and the new method can be found in *Appendix A* – on the next page. The Airy-functions are good, asymptotic solutions of the Stokes-equation, but the Stokes-equation is not a good theoretical description of the reflection in inhomogeneous media.)

2. Results and conclusions

The closed-formed solution yielded from the new theoretical method opens the way of numerical model calculations. As a first computed result, the time function and the FFT-pattern of a calculated reflected signal can be seen in *Fig. 2/a.*, for a given assumed plasma frequency profile (when the electron-density is linear) and for Dirac-excitation, in the case of inhomogeneous, anisotropic electron-plasma.

The result shows that the measure and behavior of the reflection follows the given density profile like a snapshot of the inhomogeneous conditions of the medium. The signal in the lower and the higher frequency range is similar to the forward propagating signals, whistler-type appears in the lower range, while the well-known signal form can be seen in the higher range concordant with the detected TiPP-signals (Transionospheric Pulse Pairs, [8]).

But it is clearly seen that the reflection is continuous in the complete frequency-range, while the forward propagating signal crosses the inhomogeneity (the discrete lines in the FFT-pattern are caused by the finite resolution of the numerical description of the density-profile).

Another example is shown in *Fig.2/b.*, for linear plasma frequency profile and Dirac-excitation.

As it was shown above, the presented solving method of inhomogeneous problems does not lead to differential equation of Riccati-type, but it is possible to obtain the solution in closed form.

The first approximation results in the well-known (non-coupled) W.K.B. approximation for weakly inhomogeneous medium, but the full-wave result opened the way to obtain more and more accurate solutions for the propagating signal in the case of stronger inhomogeneities as well. Moreover, it became possible to determine the reflected signal in closed form. By applying these results, it is possible to compute some phenomena – e.g. the reflected part of ELF-VLF signal generated by lightning, traveling through the ionosphere up to the magnetosphere (whistler-precursors). The solving method is applicable in other media too, different from the presented plasma model (e.g. mine detection).

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The Appendix:
"Comparison of the new model with the Airy integral" – see on the next page.

Appendix

Comparison of the new model with the Airy integral

One of the most commonly known theoretical approximations of wave-propagation in inhomogeneous media is the solution of the Stokes-equation by Airy integral functions [4]. It is useful to investigate the differences between the new method (presented in this paper and in [1]) and the Airy-solution. This comparison will be demonstrated for (longitudinally propagating) monochromatic signals.

As it can be found e.g. in [4, Chapter 9 and 15], the Airy integral (Airy integral functions) is the mathematically correct solution of a type of differential equations (like Stokes-equation is):

$$\frac{{}^2E_y}{dz^2} + k_0^2 q^2 E_y = 0 \qquad \text{(A.1)}$$

where $q^2 = n^2$ (longitudinal propagation), (A.2)
n is the refraction index, as usual.

The cornerstone of the new method based on MIBM is the realization of the physical fact, that only and exclusively the resultant sum of the forward propagating and reflected (scattered) signals can be an existing, real solution of Maxwell's equations.

As it is well seen in Budden's argumentation, the supposed starting form of the solution to be determined contains the resultant sum of forward and backward propagating signal-parts (see eq. (9.47) in [4]):

$$E_y = A \cdot \mathrm{e}^{-\mathrm{j}k_z z} + B \cdot \mathrm{e}^{\mathrm{j}k_z z} \qquad \text{(A.3)}$$

where $k_z = k_0 \cdot n = \frac{\omega}{c} \cdot n$

Further, the detailed investigation of the derivation enlightens some important problems. Budden applies Maxwell's equations, and deduces the Stokes-equation from them (eq. (9.49)–(9.54) in [4]). As he states, this should refer to the signal-form (A.3), and a result of this deduction is the known Stokes-equation (eq. (9.58) in [4]). The Airy integral functions are valid for this differential equation type.

But it is important to recognize, that Budden supposes by the introduction of the signal form of (A.3), that the substitution of the forward and backward propagating signal parts separately into Maxwell'equations results formally identical equations, and he deduces the Stokes-equation for only the forward propagating part, independently. He does not apply the resultant sum of the signal-part in his computation in order to get the Stokes-equation. This form of the Stokes-equation cannot be obtained for the resultant sum from Maxwell' equations, only for the forward or backward propagating signal-parts independently (if we suppose that one of these signal-parts can exist alone). This assumption is not a special case of the new theoretical model presented in this paper, but means a fundamental contradiction between the former approximation and the new model.

In order to compare this solving method to the new (presented in this paper and in [1]), let us control Budden's derivation. If the whole solution-form shown in (A.3) is written back into the Stokes-equation and one derives the equations on a correct way, it will be obvious, that the result presented by e.g. Budden cannot be yielded.

Assuming (in concordance with Budden's work), that A and B are constant (it must be emphasized, that this precondition is a hard restriction of the validity limits in the case of spatial inhomogeneities) and substituting (A.3) into the Stokes-equation, the following will be obtained:

$$A = -B \cdot \mathrm{e}^{\mathrm{j}2k_z z} \qquad \text{(A.4)}$$

This is in obvious and fundamental contradiction with the starting precondition (A and B have to be constant). It well can be seen, that Budden's formulas are valid only for the forward and the backward propagating signal-parts separately, so this way of thinking implicitly considers this signal-parts as independent solutions of Maxwell'equations. (A.4 cannot be considered as some "reflection coefficient", because of the starting mathematical suppositions.)

If A and B are not constant, the result does also not lead back to the known formulas, from which the Airy functions are deducible, but gives a more complicated relation between A and B:

$$\left[-2\mathrm{j}\frac{\mathrm{d}A}{\mathrm{d}z}(\mp k_0 q) - \mathrm{j}A\left(\mp k_0 \frac{\mathrm{d}n}{\mathrm{d}z}\right) + \frac{\mathrm{d}^2 A}{\mathrm{d}z^2}\right] \cdot \mathrm{e}^{-\mathrm{j}k_z z} + \left[2\mathrm{j}\frac{\mathrm{d}B}{\mathrm{d}z}(\mp k_0 q) + \mathrm{j}B\left(\mp k_0 \frac{\mathrm{d}n}{\mathrm{d}z}\right) + \frac{\mathrm{d}^2 B}{\mathrm{d}z^2}\right] \cdot \mathrm{e}^{+\mathrm{j}k_z z} = 0 \qquad \text{(A.5)}$$

This relation, on the one hand, does not coincide with results published by Budden (and others), and, on the other hand, it results an underdetermined description of the problem (one equation containing two unknown variables), which is unsolvable.

This investigation obviously confirms, that Budden's theory (which finally leads to the Stokes-equation, as a description of wave-propagation in inhomogeneous media) at least implicitly contains the preconception, that the forward and backward propagating signals are independently existing solutions of Maxwell'equations.

Because of these fundamental theoretical differences, the new method never results the Stokes-equation, but uses a new and theoretically different way in order to solve the problem of spatially inhomogeneous media. This new method delivers mathematically correct answers for the problems presented above.

Furthermore, this fact is independent from the nature of the signal (monochromatic or impulse); this argumentation is equally valid for both cases. The direction of propagation (referring to the gradient of inhomogeneity; longitudinal, transversal or oblique) has also no influence on this theoretical difference.

Summarizing, the new method never leads back to Stokes-equation during the solving process, because of this it becomes possible to avoid the theoretical contradictions presented above and to obtain an exact solution.